A Comparison of Techniques for Introducing Material Implication

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Abstract

A large volume of research shows that humans reason poorly about conditional statements and that the formal notion of material implication is difficult to learn. Textbooks on logic have used a variety of approaches to the introduction and justification of a truth-functional definition of material implication. This paper surveys eight such techniques — definition by truth table, definitions based on other logical operators, the use of examples, ways of avoiding the need for a definition, an adaptation of Peirce's notation, the motivation of universally quantified conditionals, the analogy with contractual reasoning, and finally a new suggestion based on elementary set theory.

Motivation

Material implication is an attempt to capture the essence of conditional statements: that is (in English at least), statements of the form "if ... then ...". However, statements of this form have a variety of intentions, not all of which are truth-functional. Thus, no truth-functional definition of implication will be able to fully capture the diversity of meanings in conditional statements.

A large volume of research shows that people do not naturally reason well about conditional statements. For example, the innumerable variations of Wason's Four-Card Selection Problem [see, for instance GILH88 pp113–123, EVAN82 Chapter 9 and WASO83]. Further support for this claim comes from studies with late high-school students, college students and medical students by O'Brien and Shapiro which indicate that less than 10% of subjects reason correctly about conditional statements [OBRI72, OBRI73, SHAP73]. A large part of the confusion arises from misinterpreting conditional statements as bi-conditionals. (See [FULD89] on why this mistake is so easily made.)

Furthermore, it is difficult for many students to accept the truth-functional definition of material implication. This is at least partly because it violates their intuition about conditional statements with false antecedents. To the student, it seems ludicrous to suggest that when a statement of the form "if ... then ..." has a false antecedent, the overall statement should be considered to be true. The truth table for material implication is not at all self-evident and must be accompanied by some explanation.

Approaches to Explaining Material Implication

Given the difficulties students encounter with conditional statements and with understanding the definition of material implication as a truth function, it is interesting to compare the methods used to justify this definition. There are many approaches to this — some indicate a primary concern with technical precision while others show varying degrees of concern for avoiding or alleviating student distress. After reading as many logic textbooks as I could find, I have gathered these approaches into seven categories which are described and critiqued below. An eighth approach is also suggested which I have not seen in any textbook.

1. Definition by Truth Table

Most commonly, material implication is defined by truth table [GRIM94, KOLM87, BAUM96] or some verbal equivalent such as "$X \rightarrow Y$ is always true if X is false and also if Y is true" [HILB50 p4] or "A conditional sentence is false if the antecedent is true and the consequent is false; otherwise it is true" [SUPP57 p6, see also GRASS96].

This definition is justified by the authors in various ways, though frequently no justification is given at all [HILB50, COOL42, WERK48, BELL77, ROBI79, MANN85[1], PAUL87, REEV90]. Quine claims that the given truth table "constitutes the nearest truth-functional approximation to the conditional of ordinary discourse" [QUIN40 p15] and adds that this definition dates back to Philo of Megara[2]. [AHO92], [DOWS86] and [BASS70] take the same approach, admitting that this truth function does not always match English usage. Suppes takes a bold approach and simply states that in maths and logic this is the way it is done! [SUPP57]

Shoenfield defines material implication as a function rather than as a truth table, though the effect is the same. He claims that this definition follows the "mathematical meaning of if ... then" [SHOE67 p11].

A variant on the truth table approach, shown in Figure 1, starts by showing that sixteen distinct truth tables may be constructed for expressions with two variables. After columns 1 (tautology), 16 (inconsistency), 2 (disjunction), 8 (conjunction) are discussed and named, the author draws the reader's attention to column 5 and says in effect "this is an interesting and useful column so let's give it a name as well". This approach is taken by [JEFF67 p49] and [KORF74 p254]. Jeffrey also comments "Except in odd cases the truth conditions for the indicative English conditional are accurately given by the usual truth table [i.e. Column 5 in Figure 1]" [JEFF67 pviii].

P	Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

Figure 1 — Sixteen Possible Truth Tables

2. Definition in Terms of Other Operators

Other authors define material implication as an abbreviation of some other boolean expression. The normal form of this definition is (P⇒Q) $=_{def}$ (~P∨Q) [WHIT25, STEB46, STEB52, EATO31] while others use (P⇒Q) $=_{def}$ ~(P&~Q) [QUIN41, COPI67, MITC62, CARN80, KIRW78, HOCU79]. Virtually all books show these equivalences at some point. Several texts explicitly note that the two definitions are interchangeable, and Ambrose and Lazerowitz make a major point of showing that not only can material implication be defined in terms of negation and conjunction, but one could equally well define material implication as the primitive operation and then define disjunction and conjunction in terms of negation and material implication[3] [AMBR48]. They prefer the definition (P⇒Q) $=_{def}$ ~(P&~Q) over (P⇒Q) $=_{def}$ (~P∨Q), saying that while the first makes good English sense, the second looks problematic, even though the two are provably equivalent [AMBR48 p75].

When Davis introduces the horseshoe operator ("⊃") he defines it to be equivalent to both ~(P&~Q) and (~P∨Q). Long before this, however, he discusses non-truth-functional conditional statements (using the symbols "⇒" and "→") and presents the idea of material implication in syllogistic arguments. [DAVI86]

Like the explicit definition by truth table, this form of definition is sometimes not accompanied by any justification [WHIT25, STEB46]. In her later work, Stebbing notes that what ~P∨Q defines is *material implication*, which is not necessarily the same as the English "if ... then ..." structure [STEB52 p139].

3. Definitions Relying on Examples

Several authors use selected examples (either in mathematics, science or conversational English) to justify their definition of material implication. Rosser uses some verbal trickery to show that English phrases of the form "if A then B" are the same as "we cannot have both 'A' true and 'B' false" [ROSS53, p15]. He also cites a number of mathematical examples. Massey uses the example that "If my memory is correct, then I owe you a dollar" means the same as "Either it is false that my memory is correct or else I owe you a dollar". This example suggests that P⇒Q is simply an abbreviation for ~P∨Q [MASS70 p52]. [See also GUTT71, KELL90, SAIN91, TUCK95, RUBI89, LABU93, CHUR90 and PRIO55.]

Hermes claims that the structure ~(P&~Q) occurs frequently in mathematics and that mathematicians express this as "if P then Q". With this as justification, he defines the two to be equivalent [HERM73]. Hamilton, writing specifically for mathematicians, justifies his truth table definition with the example "if $n>2$ then $n^2>4$" which, he says, is still a true statement even when *n* happens to be less than two [HAMI78 p5, see also GRIM94 p53].

Quine draws on an English example to convince the reader of his claim that "if P then Q" is equivalent to ~(P&~Q), though he does this without explicitly constructing truth tables [QUIN41 p20]. In a later work, Quine writes —

> An affirmation of the form "if P then Q" is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If on the other hand the antecedent turns out to have been false, our conditional affirmation is as if it had never been made. [QUIN82 p21]

Consequently, he claims that the choice of declaring a conditional to be true whenever the antecedent is false is *arbitrary*. Kneebone, who relies on truth tables to define material implication rather than examples, makes a similar point —

> The truth-values that are to be ascribed to $\phi\to\varphi$ in cases in which ϕ is false are unimportant, since we do not draw conclusions from premises unless these are known to be true, or at least assumed to be true for the sake of the argument; but it *greatly simplifies the formal logic* of propositions if we define the truth-value of $\phi\to\varphi$ in all cases, taking it as T whenever ϕ has the truth-value F (compare with such *conventional* definitions in mathematics as $a^0=1$ and $0!=1$). [KNEE63 p31, emphasis mine]

Mendelson also claims that the truth-functional definition is simply a convention, though it is also justified by the desire that (P&Q)⇒P should be a tautology [MEND64 p13].

Korfhage uses a pedagogically fascinating, though technically misguided analogy with computer programming to show why a conditional should be treated as true if the antecedent is false. He writes that given P⇒Q, where P is known to be false, it is true that we can't deduce anything about Q, however, we don't want the argument to stop there. Compare this with a FORTRAN program containing the statement "IF ALPHA .GT. X+7 GOTO 13". Even if "ALPHA .GT. X+7" is false, the overall statement is a good piece of FORTRAN and we want the program to continue running. Hence, so as not to disrupt an argument, we assign P⇒Q the value true whenever P is false. [KORF74]

This approach may appeal to students who already understand the FORTRAN "if" statement. However, there is some evidence that children who have been taught the "if ... then ... else" programming construct misconstrue conditional statements as bi-conditionals [SEID89]. That is, after being exposed to the program language interpretation of an "if" statement, they infer an incorrect truth table for material implication. Korfhage's approach is technically misguided since it confuses a form of conditional in which the antecedent and consequent are causally connected ("if the condition ALPHA .GT. X+7 is true then the next thing to do is execute the instruction at line 13") with the truth functional form which requires no causal connectivity.

The method of choosing or contriving an example which fits the author's intention is rather artificial. A more sophisticated approach is to note that the English "if ... then ..." structure is used in a variety of senses and that since we need to use symbolic operators unambiguously, we must choose just one of those senses. According to Church, "we select the one use of the words 'if ... then' ... in which they may be construed as denoting a relation between truth-values" [CHUR56 p38].

Reichenbach, who uses a truth table to define the horseshoe operator, makes a useful distinction between *adjunctive* implication and *connective* implication.

> It recently happened in Los Angeles that, while the screen of a movie theatre was showing a blasting of lumber jammed in a river, an earthquake shook the theatre. The implication "the blasting of lumber on the screen implied the shaking of the theatre" was then true in the adjunctive sense whereas it was false in the connective interpretation. ... We realise that the word "implies" here has not the same meaning as in conversational language; the implication in this case simply *adjoins* one statement to the other without *connecting* the statements. Adjunctive implication has a wider meaning than connective implication; if a connective implication holds, there also exists an adjunctive implication, but not vice versa. [REIC47 pp29,30]

Copi's "Introduction to Logic" has the most extensive variant of this approach. After explaining how we choose one of the two senses of the English "or" (the inclusive rather than exclusive sense), he lists four senses of implication and then chooses the one which can be written symbolically as ~(P&~Q). Choosing this interpretation over the other three is not arbitrary. Rather, he shows that this definition specifies the *common ground* between the four senses. [COPI67 pp245–252]

Galton argues that the truth table definition is the minimal truth-functional definition which will apply to all conditional statements.

> Even though $\rightarrow$ may not capture everything that is implied by "if ... then ...", at least we can say that a statement of the form $A \rightarrow B$ will be true whenever (if not more often than) "if A then B" is true, so that the English inference with statements of this form amongst its premises will be valid so long as the propositional calculus translation is. [GALT90 p57]

Georgacarakos and Smith devote many pages to this same point [GEOR79 p53ff]. Forbes discusses various linguistic issues related to conditional statements and only later constructs a truth table to fit selected English examples [FORB94].

4. Avoiding any Truth-Functional Definition

Another way to tackle the problematic definition of material implication is to avoid explicit definition altogether [FITC52, LEMM71, POSP84]. Fitch uses a natural deduction system in which conditional expressions may be manipulated by Modus Ponens and Distribution[4]. No mention is made of truth tables and $(\sim P \vee Q) \Rightarrow (P \Rightarrow Q)$ is left as an exercise for the reader [FITC52]. Lemmon follows a similar path, using the symbol $\Rightarrow$ in proofs long before defining it as a truth function. He uses a natural deduction system with ten Rules of Derivation to prove that $P \Rightarrow Q$, $\sim P \vee Q$ and $\sim(P \& \sim Q)$ are all interderivable. When he eventually gets around to discussing truth tables, it is then clear that $P \Rightarrow Q$ should be defined to have the same truth table as both $\sim P \vee Q$ and $\sim(P \& \sim Q)$. Even so, he admits that the truth table definition of material implication "seems rather arbitrary" [LEMM71 pp67–68].

5. Using Peirce's Notation[5]

Another approach relates material implication to the mathematical concept of less-than-or-equal-to. This is inspired by Peirce, who used a modified "≤" sign to stand for material implication in 1885 [HART74 paragraph 3.373]. The implication $P \Rightarrow Q$ can be explained by showing that the truth of Q is at least as certain as the truth of P, because Q must be true whenever P is true, but Q can also be true on other grounds independent of P. If the value "true" is interpreted to be greater than the value "false", then the truth table for material implication will be identical to the truth table for less-than-or-equal-to.

This explanation may be especially useful for computer science students since they will already have a mental correspondence between "true" and "false" and the binary values 0 and 1.

6. Justification Based on Universal Quantification

Although most textbooks introduce material implication well before quantification, this need not be the case. For a student already familiar with the idea that a universally quantified expression is like an infinite conjunction, the following approach may help to explain the truth table for material implication.

Suppose we have symbolised the claim that "All wolves are carnivorous" as $\forall x(W(x) \Rightarrow C(x))$. We want this claim to be true, and thus require that the expression $W(x) \Rightarrow C(x)$ be true for every possible value of x, including values of x which are not wolves. Motivated by this, we can ask ourselves what to do in the case when x is a wolf and also carnivorous, in the case when x is neither a wolf nor carnivorous, etc, and fill in the four rows of a truth table accordingly. It will be seen that even when $W(x)$ is not satisfied, the expression $W(x) \Rightarrow C(x)$ must be designated as true in order that the overall statement $\forall x(W(x) \Rightarrow C(x))$ is confirmed. [GRAN96]

7. Definition Based on the Idea of Contracts

In [NISB87], Nisbett et. al. claim that an effective way to teach conditional logic is to draw on pre-existing concepts rather than to define an entirely new concept. The pre-existing concepts they suggest are those of *permission* and *obligation*, both of which are forms of contract.

The statement "In order to do some action P you must have permission Q" follows precisely the same truth-function as $P \Rightarrow Q$. A contract of permission is violated only when the action P is performed without the required permission Q. If P is performed *with* permission Q, or if P is not performed at all, then the permission contract stands unviolated.

The statement "If you perform some action P then you are obligated to do Q" follows the same pattern. Such an obligation is violated only when P occurs but not Q, and hence an obligation schema behaves the same as material implication. This schema may also be presented in terms of *promises*, though perhaps not as effectively since a promise is typically less strictly specified than a legal obligation [LABU93, GRAN96].

Note that permission and obligation are not presented simply as examples as described in the third category above. Rather, the aim is to proffer these to students as inference schema with which they are already well acquainted, and to indicate that processing conditional statements should be undertaken using those same schema.

8. Definition Based on Elementary Set Theory

If one can assume that students have an understanding of the basic concepts of set theory, then those concepts can be matched with parallel concepts in boolean logic. Negation can be explained as the logical counterpart to set complementation; conjunction as the counterpart to intersection; disjunction as the counterpart to union[6]; material implication as the counterpart to the subset relation; and the bi-conditional as the counterpart to set equivalence. This approach does not imply that there is a formal equivalence between set expressions and propositional expressions, but rather that set operators invoke the same reasoning schema as propositional operators.

Students familiar with set concepts will understand the assertion $P \subseteq Q$ to mean that every element of P is also an element of Q. But this is a disguised form of the assertion "if x is an element of P then x is an element of Q", or symbolically, $x \in P \Rightarrow x \in Q$. Thus, an appropriate explanation of the subset relationship can provide a very clear justification of a truth-functional material implication.

Such an explanation may proceed as follows. Consider four arbitrary members of the universe x_1, x_2, x_3 and x_4 such that $x_1 \in P$ and $x_1 \in Q$, $x_2 \in P$ and $x_2 \notin Q$, $x_3 \notin P$ and $x_3 \in Q$, $x_4 \notin P$ and $x_4 \notin Q$. Three of these (x_1, x_3 and x_4) are consistent with the assertion $P \subseteq Q$, but the fourth (x_2) is inconsistent with that assertion (see Figure 2). The claim $P \subseteq Q$ can only be shown to be false by finding an object which *is a member* of P but *is not a member* of Q. By analogy, the claim $P \Rightarrow Q$ is only false in the circumstance where P *is true* but Q *is not true*. The truth table for $P \Rightarrow Q$ will have three rows (corresponding to x_1, x_3 and x_4) marked as true and one row (corresponding to x_2) marked as false.

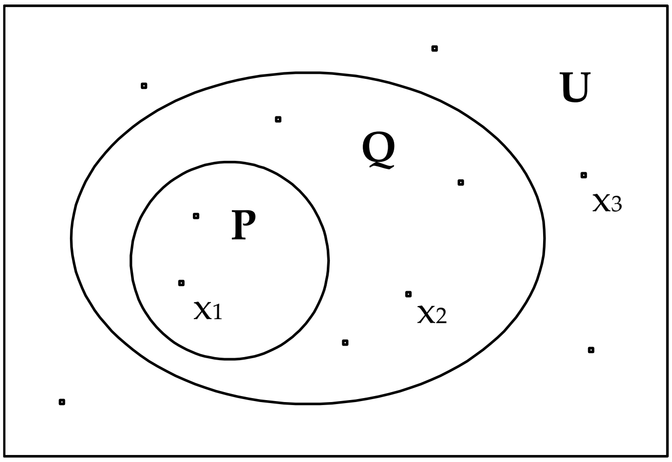

Figure 2 — What would disprove that P⊂Q?

I have not seen a textbook which explicitly links the definition of logical operators to set operators in this way, but I have found this approach effective in the classroom. The approach has several advantages —

- Since students already have a grounding in set theory, the approach defines new concepts in terms of familiar concepts.
- The visual model of set structures given by Venn Diagrams can immediately be transferred as a tool to aid understanding of logical expressions.
- The definition of material implication is no longer "arbitrary", as some of the authors quoted above apologetically assert. The truth table for material implication can be shown to follow naturally from the subset relationship P⊆Q which unambiguously *disallows* the situation where membership of P is true but membership of Q is false, and *allows* the other three possible situations.

Conclusion

When introducing the concept of material implication to students, a teacher may choose from a number of techniques. Eight such techniques have been described in this paper and we have noted that some emphasise technical correctness while others more directly promote student understanding.

Our suggestion is that defining material implication by truth table provides insufficient explanation or justification for many students. Such definitions may enable students to manipulate logical expressions correctly, but with only a surface-level understanding of what they are doing. Techniques such as definition based on the subset relationship and definition based on the idea of a contract should be used to promote deeper conceptual understanding.

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End Notes

[1] Manna and Waldinger have the added novelty of defining a truth table for "if ... then ... else"! [MANN85, p13]

[2] Peirce provides a summary of the debate between Philo and Diodorus on whether hypothetical propositions (i.e. conditionals) are at all different from categorical propositions. Philo claims (and Peirce agrees) that the forms "If P then Q" and "Every P is Q" are identical, but Diodorus (supported by Peirce's contemporary Shröder) claims they have different meanings. [HART74, paragraph 3.439ff, written in 1896]

[3] There is at least one book which actually takes this approach — [BELL77].

[4] The rule of Distribution may be symbolised as $P \Rightarrow (Q \Rightarrow R) \therefore (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$.

[5] Suggested by John Sowa <sowa@turing.pacss.binghamton.edu> pers. comm.

[6] The English word "or" is sometimes interpreted inclusively and sometimes exclusively, and when introducing logical connectives one must somehow address the issue of which way to interpret disjunction. Various approaches are used in textbooks, but a key advantage of presenting disjunction as the counterpart to union is that it provides a very natural rationale for making disjunction inclusive — just as the set $A \cup B$ includes elements which are common to both sets, so $A \vee B$ is true even when both disjuncts are true.