Possible Models Diagrams

A visual alternative to truth tables

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Abstract

Truth tables are a basic tool for teaching propositional logic. However, when students are unfamiliar with logic or come from educationally disadvantaged backgrounds, definitions of logical connectives based on truth tables are unintuitive and difficult to remember. This is especially the case with the truth table for material implication.

Possible Models Diagrams are a visual representation of propositional expressions and hence offer an alternative to truth tables. Like truth tables, Possible Models Diagrams can be used to define basic logical connectives and to classify wffs as tautological, contingent or inconsistent. In addition, they show greater expressibility than truth tables in that they can be used to show whether or not a sequent is provable. Furthermore, they are simpler to learn and to remember because of their visual appeal.

1. Introduction

It is standard practice to introduce university students to truth tables quite early in computer science courses. Truth tables provide a convenient way to teach boolean logic, which forms the basis of the digital computer. Truth tables also introduce propositional logic concepts and are often presented along with some formal proof structure and associated rules of derivation.

Later, the limitations of propositional calculus lead to the need for predicate calculus and here truth tables must be put aside. Nevertheless, the concepts described by truth tables form essential groundwork for any career in either logic or computing.

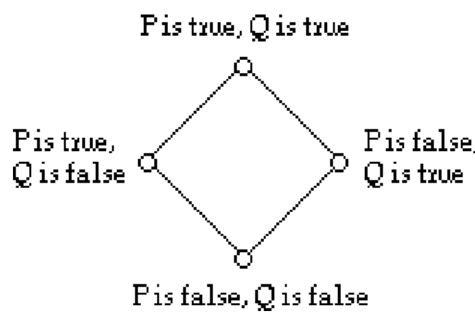
Regardless of the fact that truth tables are virtually universally standard, they aren't so sacrosanct as to make alternatives unthinkable. Indeed truth tables do present some teaching problems which allow for some improvement. In particular, the truth table definition of material implication (and disjunction to a lesser extent) always confuses students. The fact that an implication should be considered true when the antecedent is false is neither intuitive nor easily remembered.

Karnaugh maps are sometimes presented in conjunction with truth tables, but in the interest of good pedagogy, this paper presents an alternative which is as equally expressive (indeed more expressive than truth tables), yet more intuitive to the novice and more easily remembered by the student.

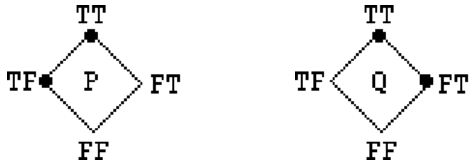
Section 2 presents a student's-eye-view of Possible Models Diagrams (PMD), and then Section 3 supports the logical soundness of the approach.

2. Possible Models Diagrams: a student guide

Suppose P and Q represent two propositions. Then there are four possible states of the world: the state in which both P and Q are true; the state where P is true but Q is false; the state where P is false but Q true; and the state where both P and Q are false. We could show this situation in a simple graph -



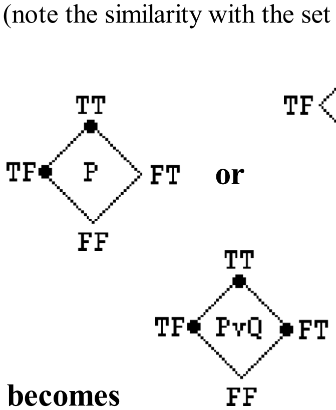
Each of these possible situations is called a model (or an interpretation) of P and Q and hence the graph is called a Possible Models Diagram (PMD). We show that some expression of propositional logic is true for a particular model by filling in the corresponding node of the graph. Thus the simple propositions P and Q are represented by the PMDs -



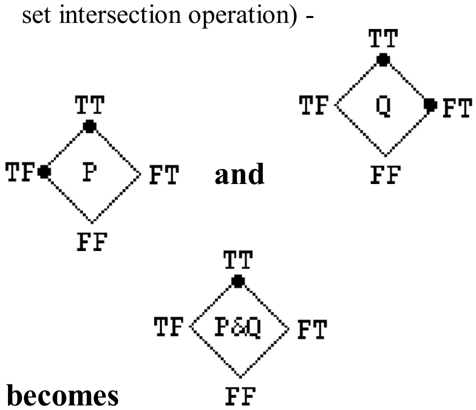
2.1 Combining rules

We have seen that Possible Model Diagrams are an easy way of visualising simple propositions and now show a number of ways of combining these diagrams to represent compound statements.

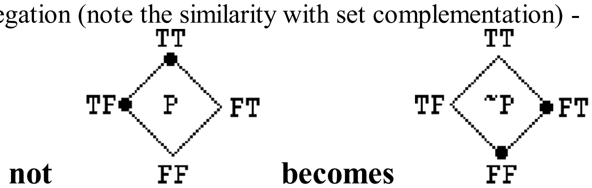
2.1.1. Overlaying two diagrams has the same effect as disjunction (note the similarity with the set union operation) -



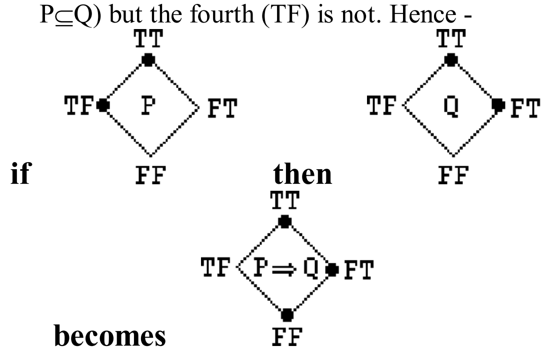
2.1.2. Matching-the-dots on two diagrams has the same effect as conjunction (note the similarity with the set intersection operation) -



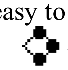
2.1.3. Reversing each corner of a diagram is the same as negation (note the similarity with set complementation) -



2.1.4. How do we capture the idea of a subset in propositional logic? Suppose we want to establish that $P \subseteq Q$ (ie that if $x \in P$ then $x \in Q$). Of the four possible models of P and Q, three are consistent with this subset relation (TT, FT and FF) (ie these three points could be plotted on a Venn Diagram where $P \subseteq Q$) but the fourth (TF) is not. Hence -



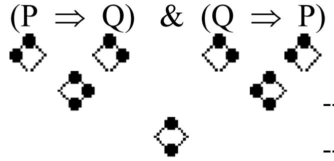
Two things to note about this diagram -

i) It is easy to remember that implication has the diagram  because the three dots look like an arrow pointing to the right.
ii) When combining two diagrams as above, the visual procedure is to reverse the first diagram and overlay it on the second[1].

2.2 More complex expressions

The equivalence relation for sets $P \equiv Q$ is reflected in propositional logic by the bi-conditional: $P \Leftrightarrow Q$ is defined as $(P \Rightarrow Q) \& (Q \Rightarrow P)$. This can be diagrammed by appropriate combinations of the previous operations -

$(P \Rightarrow Q) \quad \& \quad (Q \Rightarrow P)$



-- Reverse and overlay

-- Match-the-dots

The final outcome is intuitively sensible, since it shows those models in which the truth of P is exactly the same as the truth of Q.

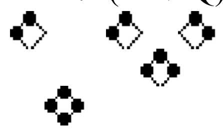
In fact a PMD can be constructed using the simple combining rules (reverse, overlay and match-the-dots) for any well-

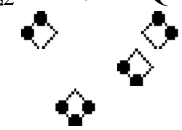
[1] To the student, the procedure "reverse and overlay" is an easy habit to learn and it is only later that they discover that $P \Rightarrow Q$ is logically equivalent to $\sim P v Q$. Some teachers may prefer to use this equivalence as a definition of material implication, in the same way that Copi [1] uses $\sim(P \& \sim Q)$ as a definition. I find that students rebel against this use of fiat, but are more ready to accept the parallel between implication and subset.

Lemmon [2] avoids this difficulty by never mentioning truth tables until the very end of his coverage of propositional calculus. In that way he can show how $\Rightarrow$ works in formal proofs well before he is forced to define it.

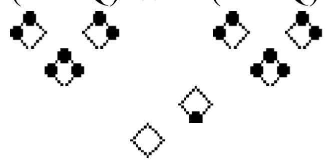
formed formulae which uses only two propositional variables. For example -

eg$_1$ $P \Rightarrow (P \vee Q)$ eg$_2$ $P \Rightarrow \sim Q$





eg$_3$ $(P \vee Q)\ \&\ \sim (P \vee Q)$

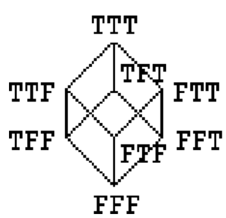


Every wff falls into one of three categories -

i) Tautology - a wff whose PMD has every corner filled in (eg$_1$). ie a wff which is true however you interpret it.
ii) Contingent - a wff whose PMD has some corners filled in but not all (eg$_2$). ie a wff which is sometimes true and sometimes false.
iii) Inconsistent - a wff whose PMD has no corner filled in (eg$_3$). ie it is always false.

2.3 Generalising to Three Variables

If there are three propositional variables (say P,Q,R) in a wff, there will be eight possible models. These are best visualised as the corners of a cube whose opposing faces represent P and ~P, Q and ~Q, R and ~R respectively. However, this can be shown diagrammatically as the following two-dimensional graph -



The combining rules (reverse, overlay and match-the-dots) can all be used as before.

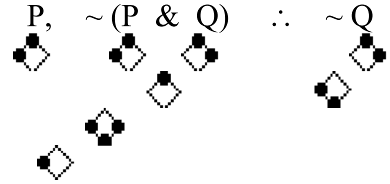
2.4 Provable sequents

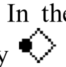
After showing how complex statements may be represented as symbolic propositional formulae, a course in logic is then likely to proceed to the concept of a structured argument. We describe a sequent in some form like $A_1,A_2,A_3,...,A_n \therefore B$ (meaning "the premises $A_1,A_2,A_3,...,A_n$ entail the conclusion B") and then describe a range of derivation rules which allow one to proceed logically from one statement to the next in a proof of the sequent.

In any propositional calculus, a sequent $A_1,A_2,A_3,...,A_n \therefore B$ may be shown to be valid (and hence provable) by showing that $(A_1 \Rightarrow (A_2 \Rightarrow (A_3 \Rightarrow ...(A_n \Rightarrow B)...)))$ is a tautology, however, this is rather cumbersome in general. PMDs provide an alternate way to check whether a sequent is valid.

Suppose we want to test whether the sequent P, ~(P&Q) $\therefore$ ~Q is valid. First, construct a PMD for the wffs on the left-hand side (the comma is taken to be an implicit conjunction) and a separate diagram for the right-hand side -

$P,\ \sim(P\ \&\ Q)\ \ \therefore\ \ \sim Q$



Now apply this simple rule: **a sequent is valid if and only if the diagram for the premises is a subset of the diagram for the conclusion**. In the example above, the possible models represented by  are a subset of those represented by  and hence the sequent is valid.

3. Possible Models Diagrams: logical soundness

3.1 Possible Models Diagrams are really sets

In order to see that PMDs are a sound way of representing wffs, one has to first forget the lines connecting the graph and simply think of the nodes as the set of all possible models of n propositions $U_n = \{<x_1,x_2,...x_n> \mid x_i \in \{T,F\}\}$. The lines link elements of this set in a certain way, but this is only important for the visual effect.

For n=2, we get the set of all possible models of two propositions $U_2 = \{<T,T>,<T,F>,<F,T>,<F,F>\}$. Now we only need to equate the first proposition P with the set $\{<x,y> \in U_2 \mid x=T\}$ and the second proposition Q with the set $\{<x,y> \in U_2 \mid y=T\}$.

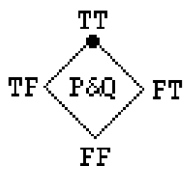
Suppose W is the set representing some wff. Then the reverse operation is simply $\sim W = \{<x,y> \in U_2 \mid <x,y> \notin W\}$. If V is a set representing some other wff, then overlay corresponds to $V \cup W = \{<x,y> \in U_2 \mid <x,y> \in V \text{ or } <x,y> \in W\}$ and match-the-dots corresponds to $V \cap W = \{<x,y> \in U_2 \mid <x,y> \in V \text{ and } <x,y> \in W\}$.

Thus, the operations defined on PMDs correspond to exactly those primitive logical operations we expect.

3.2 Converting between Truth Tables and Possible Models Diagrams

Whereas a PMD represents all possible models as nodes on a graph, a truth table represents them as lines in a table. Given a PMD, it is simple to construct an equivalent truth table: for each node in the diagram, if the node is filled-in, place a T in the corresponding row of the table, otherwise place an F in the corresponding row of the truth table. Converting in the opposite direction is equally trivial. For example -

3.3 Generalising to more than two propositional variables



$\Leftrightarrow$

P	Q	P & Q
T	T	T
T	F	F
F	T	F
F	F	F

Given that a PMD is just a way of visualising the set of possible models (Section 3.1), it should be clear that such diagrams can be formed for wffs containing any number of propositional variables. In general, if a wff contains n distinct propositional variables, then its PMD will be an n-cube with 2^n nodes (just as the corresponding truth table will have 2^n rows).

In teaching situations it is rare to set a truth table problem with more than three variables, and the same would apply to PMDs. Both methods in theory generalise, but in practice we would rarely use either method for cases where the number of variables is large.

3.4 The rule for sequent validity

In Section 2.4 we saw the rule “a sequent is valid if and only if the diagram for the premises is a subset of the diagram for the conclusion.” This can be justified by again thinking in terms of sets.

Suppose the conjunction of premises forms the wff X, represented by a PMD whose nodes form the set $X' \subseteq U_n$, and suppose the conclusion Y gives rise to a Diagram whose nodes form the set $Y' \subseteq U_n$. If $X' \subseteq Y'$ then every possible model of the premises is also a model of the conclusion.

Now if every possible model for which some condition X is true is also a model for which Y is true, then X is a sufficient condition for Y. Hence $X \Rightarrow Y$ is necessarily true. Further, $\therefore X \Rightarrow Y$ and $X \therefore Y$ are equivalent and so from the fact that one PMD is a subset of a second, we can deduce that the wff represented by the second is derivable from the wff represented by the first.

Conversely, if Y can be derived from X (ie $X \therefore Y$ is a valid sequent), then it must be that $X \Rightarrow Y$ is a tautology. This is the case exactly when ~XvY is also a tautology. Now according to our set interpretation (Section 3.1), this is the same as saying that the set $\{<x_1,x_2,...x_n> \in U_n \mid <x_1,x_2,...x_n> \notin X' \text{ or } <x_1,x_2,...x_n> \in Y'\}$ is equivalent to U_n. But this equivalence is only possible when $X' \subseteq Y'$.

4. Pedagogical Comments

The use of PMDs in our first year computer science course has shown reasonable success. It was felt in our department that even if PMDs were used to introduce logical concepts, we would still teach truth tables as well because of their prevalence in the computing discipline.

Admittedly the sample size is small (65 students) and the statistical analysis informal, but students seemed to grasp the concepts easily and when given the choice preferred to use PMDs rather than truth tables. In exam conditions, only 11% chose to use truth tables to analyse expressions with two variables, though 46% chose truth tables for a more complicated three-variable example. Not only did most people choose to use PMDs, but those who did had a lower error rate (18%) than those who used truth tables (58%).

Two pedagogical issues need to examined by anyone thinking of teaching with PMDs. First, one must question whether the logical concepts underlying truth tables are likely to be so difficult for students to understand and remember that an alternative approach is worth considering. The visual approach is certainly more appealing to some students, but if they must still learn truth tables, is the extra effort worthwhile? In many situations this will not be the case, but PMDs will be particularly useful when trying to teach logical concepts to students with poor educational backgrounds (as is often the case in developing countries).

Second, one must be careful that the students do not learn the visual manipulations (reverse, overlay and match-the-dots) simply as syntactic operations. The semantic reasons behind these operations must be repeatedly stressed so that the logical concepts behind PMDs are clearly understood. (This applies equally to teaching with truth tables.) One way of ensuring this is to frequently ask students to translate PMDs into appropriate English propositions and vice-versa.

5. Conclusion

Although truth tables are the standard way of introducing boolean logic, a more visual teaching tool has some advantages. This paper has presented the Possible Models Diagrams as such a visual tool. PMDs are simple to learn and manipulate and hence more enjoyable and memorable for students.

Theoretically, nothing is lost by using Possible Models Diagrams rather than truth tables: they can both be used to define basic logical operations and to test whether a wff is tautologous, contingent or inconsistent. Furthermore, Possible Models Diagrams can also be used for a task which is difficult for truth tables, namely to check the validity of a sequent.

References

1. I.M.Copi, Introduction to Logic, MacMillan, 1982
2. E.J.Lemmon, Beginning Logic, Nelson's University Paperbacks, 1971.